Algebr Represent Theor (2010) 13:335–357
DOI 10.1007/s10468-008-9126-1

Cluster Algebras and Grassmannians of Type G_2

Sachin Gautam

Received: 9 November 2008 / Accepted: 17 December 2008 / Published online: 13 February 2009

Abstract We prove a conjecture of Geiss, Leclerc and Schröer, producing cluster algebra structures on multi-homogeneous coordinate ring of partial flag varieties, for the case G_2. As a consequence we sharpen the known fact that coordinate ring of the double Bruhat cell G^{e,w_0} is an upper cluster algebra, by proving that it is a cluster algebra.

Keywords Cluster algebras · Based affine space · Exceptional group

Mathematics Subject Classifications (2000) Primary 16S99 · Secondary 05E99 · 14M15

1 Introduction

The theory of cluster algebras was founded and developed by Sergey Fomin and Andrei Zelevinsky (see [2, 6, 7]) in order to develop a framework for dual canonical basis. Since their first appearance, cluster algebras have found their place in several interesting branches of mathematics (e.g., Poisson geometry, integrable systems, Teichmüller spaces, representations of finite dimensional algebras, tropical geometry,...).

The results of this note arose from an attempt to understand the cluster algebra structures coming from representations of the group of type G_2. The link between the representations of simple Lie group G and cluster algebras has already been made in [2] where authors give a combinatorial procedure to get a cluster algebra structure on the coordinate rings of double Bruhat cells, in particular on G^{e,w_0} which is the one

Presented by Alain Verschoren.

S. Gautam (✉)
Department of Mathematics, Northeastern University, Boston MA 02115, USA
e-mail: gautam.s@neu.edu

we will be focusing on in this note. (I also learnt from Prof. Andrei Zelevinsky that the computations of dual canonical basis for the case G_2 was a major motivation for defining cluster algebras).

The starting point of this project was the algebra of N^- invariant functions on G which serves as a model for all finite dimensional representations of G, and is equipped with cluster algebra structure. In the case of G_2 there are two fundamental representations and the algebra at hand is a direct sum of finite dimensional representations parametrized by non-negative elements of a two dimensional lattice (weight lattice, generated by fundamental weights ω_1, ω_2).

$$\mathcal{A} = \bigoplus_{m,n\in\mathbb{Z}_{\geq 0}} V_{m\omega_1+n\omega_2}$$

The following questions arise naturally in pursuit of this study:

(A) For each $i = 1, 2$ we have subalgebras $\mathcal{A}_i = \oplus_{n\in\mathbb{Z}_{\geq 0}} V_{n\omega_i}$ of $\mathcal{A}$. Is there a natural cluster algebra structure on $\mathcal{A}_i$?
(B) Is it possible to extract $\mathcal{A}_i$ from $\mathcal{A}$ by elementary operations (mutations, freezing of vertices, etc.) which will make the inclusion $\mathcal{A}_i \subset \mathcal{A}$ explicit?

In [8] the authors develop a connection between cluster algebra structures on multi-homogeneous rings of partial flag varieties and representations of preprojective algebra of an underlying Dynkin quiver, for simply laced cases. They also provide a purely combinatorial algorithm to extract these structures for non-simply laced cases (and hence giving a conjectural answer to (A)), but since the techniques of [8] are only applicable in simply-laced cases, the construction remained conjectural. The construction of cluster algebra structures on N (maximal unipotent group), using representation theory of preprojective algebras, was further extended to non simply-laced cases in [4]. In loc. cit. author uses the folding process (of V. Kac and G. Lusztig) to reduce the problem to simply laced cases.

The aim of this note is to answer these two questions for the case G_2. We produce cluster algebra structures on $\mathcal{A}_i$ starting from that of $\mathcal{A}$ by "elementary operations". It turns out that the two structures are "transpose" of each other (thus $\mathcal{A}_2$ is of type $G_2^{(1)}$ and $\mathcal{A}_1$ is of type $D_4^{(3)}$). Both these structure fall into the realm of [3]. In fact the statements of Section 6 are easy consequences of results of loc. cit. and are given here just for the sake of completeness. Next we prove that our construction agrees with that of [8]. This suggests an alternative method of constructing cluster algebra structures on multi-homogeneous coordinate rings of partial flag varieties, but I do not have the answer to this question in general.

The structure of this note is as follows. In Section 2 we define the initial seed for each $\mathcal{A}_i$ and state the main theorem. We also observe that the statement of our main theorem (Theorem 2.1) implies Conjectures 10.4 and 9.6 of [8] in the G_2 case; and proves that the upper cluster algebra structure on G^{e,w_0} given in [2] is in fact a cluster algebra structure.

Section 3 is devoted to recall basic definitions from the theory of cluster algebras. In particular we recall the (upper) cluster algebra structure on G^{e,w_0} as given in [2].

In Section 4 we explain how to get the initial seeds for $\mathcal{A}_i$ starting from the initial seed of $\mathcal{A}$ and using some elementary operations of mutations and freezing of vertices. This allows us to use the results of [2] in our situation.

Sections 5, 6 and 7 are devoted to the proof of Theorem 2.1.

In Section 8 we prove that our cluster algebra structure agrees with the one given in [8]. In particular we give the explicit sequence of mutations which relates our initial seed to the one given in [8].

2 Main Results

Let G be a simple algebraic group of type G_2 (over $\mathbb{C}$) and $\mathfrak{g}$ be its Lie algebra. Let $B^{\pm}$ be a pair of opposite Borel subgroups, $N^{\pm}$ be their respective unipotent radicals. Fix $\{f_1, f_2, h_1, h_2, e_1, e_2\}$ Chevalley generators of $\mathfrak{g}$. Let $x_i(t) = \exp te_i$ and $y_i(t) = \exp tf_i$ (for $i = 1, 2$) be one parameter subgroups corresponding to Chevalley generators e_i and f_i. For each $i = 1, 2$ let P_i^- be the parabolic subgroup generated by B^- and $x_i(t)$. Consider the following multi-homogeneous coordinate rings

$$\mathcal{A}_i = \mathbb{C}[P_j^- \backslash G]$$

where $i \in \{1, 2\}$ and $j = \{1, 2\} \setminus \{i\}$. This notation will be retained throughout this note.

Let ω_1, ω_2 be fundamental weights of the root system of type G_2. It is well known that left N^- invariant regular functions on G is a model for all irreducible finite dimensional representations of G, i.e,

$$\mathbb{C}[N^- \backslash G] = \bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} V_{m\omega_1 + n\omega_2} \tag{1}$$

as right G-modules. Here we use the notation V_λ for the (unique) irreducible, finite dimensional G module of highest weight λ (where λ is some dominant integral weight). With respect to this model, we have

$$\mathcal{A}_i = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_{n\omega_i}$$

The main result of this note provides a cluster algebra structure on $\mathcal{A}_i$.

2.1 Recollections from Representation Theory

We have a left and right action of $\mathfrak{g}$ on $\mathbb{C}[G]$. In order to distinguish the two actions, we write $X^\dagger$ to denote X acting on the right.

$$(X^\dagger f)(x) = \frac{d}{dt} f(x \exp tX)|_{t=0}$$

In particular $e_i^\dagger$ will be referred to as "raising operators" and $f_i^\dagger$ will be referred to as "lowering operators". We define certain left N^- invariant functions on G, called generalized minors. Following [5], let $G_0 = N^- H N^+$ be the dense subset of G consisting of elements which admit Gaussian decomposition. For $x \in G_0$ we write $x = [x]_-[x]_0[x]_+$ where $[x]_\pm \in N^\pm$ and $[x]_0 \in H$. For each i define

$$\Delta^{\omega_i}(x) = [x]_0^{\omega_i}$$

where ω_i is thought of as a character of torus H. Finally one chooses a lifting of W to G by

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

here $\varphi_i : SL_2(\mathbb{C}) \to G$ is given by:

$$\varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_i(t),\ \ \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = y_i(t)$$

For any $w \in W$, choose a reduced expression $w = s_{i_1} \dots s_{i_l}$ and set

$$\overline{w} = \overline{s_{i_1}} \dots \overline{s_{i_l}}$$

Finally define the function $\Delta_{\omega_i, w\omega_i}$ as

$$\Delta_{\omega_i, w\omega_i}(x) = \Delta^{\omega_i}(x\overline{w})$$

Since for most of this note we will be concerned only with minors of the form $\Delta_{\omega_i, w\omega_i}$ we will denote them by $\Delta^{w\omega_i}$. It is easy to check that each of these functions is left N^- invariant and under the isomorphism (1) we have

$$\Delta^{w\omega_i} \in V_{\omega_i}(w\omega_i)$$

Alternatively, in the G_2 case, one can define $\Delta^{w\omega_i}$ as follows:

$$\Delta^{s_1\omega_1} = f_1^{\dagger} \Delta^{\omega_1} \tag{2}$$

$$\Delta^{s_2s_1\omega_1} = f_2^{\dagger} \Delta^{s_1\omega_1} \tag{3}$$

$$\Delta^{s_1s_2s_1\omega_1} = \frac{1}{2}(f_1^{\dagger})^2 \Delta^{s_2s_1\omega_1} \tag{4}$$

$$\Delta^{s_2s_1s_2s_1\omega_1} = f_2^{\dagger} \Delta^{s_1s_2s_1\omega_1} \tag{5}$$

$$\Delta^{w_0\omega_1} = f_1^{\dagger} \Delta^{s_2s_1s_2s_1\omega_1} \tag{6}$$

$$\Delta^{s_2\omega_2} = f_2^{\dagger} \Delta^{\omega_2} \tag{7}$$

$$\Delta^{s_1s_2\omega_2} = \frac{1}{6}(f_2^{\dagger})^3 \Delta^{s_2\omega_2} \tag{8}$$

$$\Delta^{s_2s_1s_2\omega_2} = \frac{1}{2}(f_2^{\dagger})^2 \Delta^{s_1s_2\omega_2} \tag{9}$$

$$\Delta^{s_1s_2s_1s_2\omega_2} = \frac{1}{6}(f_1^{\dagger})^3 \Delta^{s_2s_1s_2\omega_2} \tag{10}$$

$$\Delta^{w_0\omega_2} = f_2^{\dagger} \Delta^{s_1s_2s_1s_2\omega_2} \tag{11}$$

2.2 Initial Seeds for $\mathcal{A}_i$

In this section we define two initial seeds (see Section 3 for terminology)

Definition of $\underline{\Sigma_1}$ The exchange matrix is encoded in the following valued graph.

$$
\begin{tikzcd}
\underline{\Gamma_1}: & \mathbf{-1} \arrow[r] & 1 \arrow[r] \arrow[d] & 3 \arrow[d] & \mathbf{-3} \arrow[l, shift left] \arrow[l, shift right] \\
& & \mathbf{-2} \arrow[r] & 2 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The notations of this quiver (and all that we will consider in this paper) are as follows. The bold faced vertices are frozen (correspond to coefficients in the cluster algebra). The arrows between the top row and the bottom row are valued: if k is a vertex in the top row and l is a vertex in bottom row, then $k \to l$ means: $b_{kl} = 3$ and $b_{lk} = -1$. Therefore, in this case, we have the following exchange matrix $\underline{B_1}$:

	1	2	3
1	0	−3	1
2	1	0	−1
3	−1	3	1
−1	1	0	0
−2	−1	1	0
−3	0	−3	2

where columns are labeled by 1, 2, 3 and rows are labeled by 1, 2, 3, −1, −2, −3.
In order to set the functions we define

$$X_0 = \frac{1}{2} f_1^\dagger f_2^\dagger f_1^\dagger \Delta^{\omega_1}$$

(this is the zero weight vector in the first fundamental representation). Let

$$X_{-1} = \Delta^{w_0\omega_1}$$

$$X_{-2} = X_0(\Delta^{\omega_1}\Delta^{w_0\omega_1} + \Delta^{s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1}) - \Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1} - \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1}\Delta^{w_0\omega_1}$$

$$X_{-3} = \Delta^{\omega_1}$$

$$X_1 = \Delta^{s_2s_1s_2s_1\omega_1}$$

$$X_2 = \Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}$$

$$X_3 = \Delta^{s_2s_1\omega_1}$$

In the terminology of the theory of cluster algebras, we fix the ground ring to be $\mathbb{Z}[X_{-1}, X_{-2}, X_{-3}^{\pm}]$ (see Section 3 for definitions).

Definition of $\underline{\Sigma_2}$ The exchange matrix is encoded in the following valued graph

$$\underline{\Gamma_2}: \quad \begin{tikzcd} & \mathbf{-2} \arrow[r] & 2 \arrow[dl] \arrow[dr] & \\ \mathbf{-1} \arrow[r] & 1 \arrow[u] \arrow[r] & 3 \arrow[u] & \mathbf{-3} \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

where the notation remains the same, as in the previous part. In this case we have the following exchange matrix, $\underline{B_2}$:

	1	2	3
1	0	−1	1
2	3	0	−3
3	−1	1	0
−1	1	0	0
−2	−3	1	0
−3	0	−1	2

where columns are labeled by $\{1, 2, 3\}$ and rows by $\{1, 2, 3, -1, -2, -3\}$. Finally we set the functions

$$\begin{aligned} Y_{-1} &= \Delta^{w_0\omega_2} \\ Y_{-2} &= (2 f_1^\dagger f_2^\dagger - f_2^\dagger f_1^\dagger) Y_2 \\ Y_{-3} &= \Delta^{\omega_2} \\ Y_1 &= \Delta^{s_1 s_2 s_1 s_2 \omega_2} \\ Y_2 &= \frac{1}{6} (f_1^\dagger)^2 f_2^\dagger \Delta^{\omega_2} \\ Y_3 &= \Delta^{s_1 s_2 \omega_2} \end{aligned}$$

Again we fix the ground ring to be $\mathbb{Z}[Y_{-1}, Y_{-2}, Y_{-3}^{\pm}]$.

Remark 2.1 The choice of the functions corresponding to cluster variables will be explained in Section 4.

Remark 2.2 We will see in Section 8 that our initial seed(s) and the ones given in [8] give rise to isomorphic cluster algebras. This fact combined with Theorem 2.1 will give an affirmative answer to Conjecture 10.4 of [8].

2.3 Statements of Main Results

Theorem 2.1 *The cluster algebra associated to the initial seed $\underline{\Sigma_i}$ is isomorphic to the localization of $\mathcal{A}_i$ at multiplicative subset $S_i = \{(\Delta^{\omega_i})^n\}_{n\geq 0}$.*

$$\mathcal{A}(\underline{\Sigma_i})_{\mathbb{C}} = S_i^{-1} \mathcal{A}_i$$

We note a few corollaries to this theorem.

Corollary 2.1 *The upper cluster algebra structure on* $\mathbb{C}[G^{e,w_0}]$ (*as given in* [2], *see Section* 3 *for definition*) *is in fact a cluster algebra structure.*

As remarked in [8] (see discussion following Conjecture 10.4), an affirmative statement of Theorem 2.1 will imply the following (Conjecture 9.6 of [8]):

Corollary 2.2 *The projection* $\mathcal{A}_i \to \mathbb{C}[N_i]$ *induced from the inclusion* $N_i \to P_j^- \backslash G$ *(where* $\{j\} = \{1, 2\} \setminus \{i\}$*) defines a cluster algebra structure on* $\mathbb{C}[N_i]$.

3 Recollections from [2]

In this section we will recall some results from [2]. We will mainly focus on description of the initial seed for the coordinate rings of double Bruhat cells, in terms of double reduced words.

3.1 Definitions from Theory of Cluster Algebras

The definitions we recall here are not the most general ones. For the general theory of cluster algebras, the reader is referred to the papers [2, 6] and [7].

Let $m \geq n$ be two non-negative integers. Let $\mathcal{F} = \mathbb{Q}(u_1, \ldots, u_m)$ be the field of rational functions in m variables. A *seed* in $\mathcal{F}$ is a tuple $\Sigma = (\widetilde{\mathbf{x}}, \widetilde{B})$ where

- $\widetilde{\mathbf{x}} = \{x_1, \ldots, x_m\}$ is a set of m free generators of $\mathcal{F}$ over $\mathbb{Q}$.
- $\widetilde{B} = (b_{ij})$ is $m \times n$ matrix with integer entries, such that the square submatrix $B = (b_{ij})_{1 \leq i, j \leq n}$ obtained by taking the first n rows (called principal part of $\widetilde{B}$) is skew symmetrizable (i.e, there exists a diagonal matrix D with positive diagonal entries, such that DB is skew symmetric).

For each k, $1 \leq k \leq n$, we define the *mutation of* Σ *in the direction* k as another seed $\Sigma' = \mu_k(\Sigma)$ if $\Sigma' = (\widetilde{\mathbf{x}}', \widetilde{B}')$ where

- $\widetilde{\mathbf{x}}' = (x'_1, \ldots, x'_m)$ are given by

$$x'_i = x_i \text{ if } i \neq k$$

$$x_k x'_k = \Pi_{i=1}^m x_i^{[b_{ik}]_+} + \Pi_{i=1}^m x_i^{[-b_{ik}]_+}$$

  here we use the notation $[b]_+ = \max(0, b)$.
- The matrix $\widetilde{B}' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if} \quad i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

It is well known (and easy to check) that a) the principal part of $\widetilde{B}'$ is again skew-symmetrizable; and b) mutation is an involution (i.e, $\mu_k(\mu_k(\Sigma)) = \Sigma$). This allows us to define an equivalence relation: two seeds Σ and Σ' in $\mathcal{F}$ are *mutation equivalent* if there exists a sequence $(k_1, \ldots, k_l)$ such that $\Sigma' = \mu_{k_1} \ldots \mu_{k_l}(\Sigma)$.

If $\Sigma = (\widetilde{\mathbf{x}}, \widetilde{B})$ is a seed of $\mathcal{F}$, then the set $\widetilde{\mathbf{x}}$ is called an *extended cluster*, $\mathbf{x} = \{x_1, \ldots, x_n\}$ is called a cluster and x_i, $1 \leq i \leq n$ are called *cluster variables*.

Note that by applying mutations we only change the functions x_i $1 \leq i \leq n$. Therefore if Σ is mutation equivalent to Σ', then $x_j = x'_j$ for every $n < j \leq m$. We refer to x_j with $n < j \leq m$ as *coefficients*. We fix the ground ring R as some subring of $\mathbb{Z}[x_j^{\pm} : n < j \leq m]$ containing $\mathbb{Z}[x_j : n < j \leq m]$, i.e,

$$\mathbb{Z}[x_{n+1}, \ldots, x_m] \subseteq R \subseteq \mathbb{Z}[x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$$

Define *the upper cluster algebra* $\overline{\mathcal{A}(\Sigma)}$ as the subring of $\mathcal{F}$ consisting of functions $f \in \mathcal{F}$ such that for every $\Sigma' \sim \Sigma$, f can be expressed as Laurent polynomial in cluster variables of Σ' with coefficients from R.

We define cluster algebra associated to seed Σ, denoted by $\mathcal{A}(\Sigma)$ as the R-subalgebra of $\mathcal{F}$ generated by all cluster variables belonging to seeds $\Sigma' \sim \Sigma$. The following result is known as the *Laurent phenomenon*

$$\mathcal{A}(\Sigma) \subset \overline{\mathcal{A}(\Sigma)}$$

We say that a seed $\Sigma = (\widetilde{\mathbf{x}}, \widetilde{B})$ is acyclic if the directed graph with vertex set $\{1, \ldots, n\}$ and arrow $k \to l$ if and only if $b_{kl} > 0$, has no oriented cycles. It is known that if in the mutation equivalence class of Σ, there is some acyclic seed and the matrix $\widetilde{B}$ has full rank, then $\mathcal{A}(\Sigma) = \overline{\mathcal{A}(\Sigma)}$.

Similarly we say that seed Σ is bipartite if the directed graph defined by B as above is bipartite.

3.2 Double Bruhat Cells and Cluster Algebras

Given a pair of elements $u, v \in W$, the double Bruhat cell $G^{u,v}$ is defined as

$$G^{u,v} = BuB \cap B^- vB^-$$

In [2] the authors give a combinatorial way to construct a seed Σ of $\mathcal{F} = \mathbb{C}(G^{u,v})$, starting from a double reduced word of pair $(u, v) \in W \times W$. We will state the results of [2] regarding coordinate rings of double Bruhat cells, only in our context.

We again let G to be a simple algebraic group over $\mathbb{C}$ of type G_2. Let w_0 be the longest element of Weyl group of G. The relevant double Bruhat cell is

$$G^{e,w_0} = B \cap B^- w_0 B^-$$

As an application of Proposition 2.8 of [2] we have

Proposition 3.1 *The subset G^{e,w_0} is defined in G by the following (recall the definitions of generalized minors from Section 2.1):*

$$\Delta_{u\omega_i,\omega_i} = 0, \ u \neq e$$

$$\Delta_{\omega_i,\omega_i} \neq 0, \quad \Delta_{\omega_i,w_0\omega_i} \neq 0$$

The inclusion $G^{e,w_0} \to G$ composed with the projection onto $N^-\backslash G$ allows us to consider $\mathbb{C}[N^-\backslash G]$ as a subalgebra of $\mathbb{C}[G^{e,w_0}]$. In other words, $\mathbb{C}[G^{e,w_0}]$ is isomorphic to the localization of $\mathbb{C}[N^-\backslash G]$ at functions Δ^{ω_i} and $\Delta^{w_0\omega_i}$.

We have two reduced words for w_0, namely $\mathbf{i_1} = (1,2,1,2,1,2)$ and $\mathbf{i_2} = (2,1,2,1,2,1)$. Each one of them defines an initial seed in $\mathbb{C}(G^{e,w_0})$ (see Section 2 of [2]):

Definition of Σ_1 The exchange matrix $\widetilde{B}_1$ is encoded by the following valued quiver

$$\begin{tikzcd}
\Gamma_1 : & \mathbf{-1} \arrow[r] & 1 \arrow[r] \arrow[dl] & 3 \arrow[r] \arrow[dl] & \mathbf{5} \arrow[dl] \\
& \mathbf{-2} \arrow[r] & 2 \arrow[r] \arrow[u] & 4 \arrow[r] \arrow[u] & \mathbf{6}
\end{tikzcd}$$

The functions corresponding to cluster variables are:

$$X_{-1} = \Delta^{w_0\omega_1},\ X_1 = \Delta^{s_2s_1s_2s_1\omega_1},\ X_3 = \Delta^{s_2s_1\omega_1},\ X_5 = \Delta^{\omega_1}$$

$$X_{-2} = \Delta^{w_0\omega_2},\ X_2 = \Delta^{s_2s_1s_2\omega_2},\ X_4 = \Delta^{s_2\omega_2},\ X_6 = \Delta^{\omega_2}$$

Definition of Σ_2 The exchange matrix $\widetilde{B}_2$ is encoded by the following valued quiver

$$\begin{tikzcd}
\Gamma_2 : & \mathbf{-1} \arrow[r] & 2 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & \mathbf{6} \\
& \mathbf{-2} \arrow[r] & 1 \arrow[r] \arrow[ul] & 3 \arrow[r] \arrow[ul] & \mathbf{5} \arrow[ul]
\end{tikzcd}$$

The functions corresponding to the cluster variables are:

$$X_{-1} = \Delta^{w_0\omega_1},\ X_2 = \Delta^{s_1s_2s_1\omega_1},\ X_4 = \Delta^{s_1\omega_1},\ X_6 = \Delta^{\omega_1}$$

$$X_{-2} = \Delta^{w_0\omega_2},\ X_1 = \Delta^{s_1s_2s_1s_2\omega_2},\ X_3 = \Delta^{s_1s_2\omega_2},\ X_5 = \Delta^{\omega_2}$$

We have the following special case of Theorem 2.10 of [2]:

Theorem 3.1 *The upper cluster algebra $\overline{\mathcal{A}(\Sigma_i)}_{\mathbb{C}}$ is isomorphic to $\mathbb{C}[G^{e,w_0}]$. In particular every function obtained by applying a sequence of mutations to Σ_i is a regular function on G^{e,w_0}.*

Remark 3.1 It can be shown that the upper cluster algebras $\overline{\mathcal{A}(\Sigma_i)}$ are in fact *equal*, i.e, Σ_1 is mutation equivalent to Σ_2 by the following sequence of mutations:

$$\Sigma_1 = \mu_1\mu_3\mu_2\mu_1\mu_2\mu_4\mu_2\mu_1\mu_2\mu_3\mu_1(\Sigma_2)$$

In order to get equality above, one needs to relabel the cluster variables after the sequence of mutations. This can be verified by using quiver mutation software of Prof. B. Keller and verifying the determinantal identities for G_2 given in [1]. This result will not be used in this paper.

Remark 3.2 It will follow from our main results that we have the equality

$$\mathcal{A}(\Sigma_i) = \overline{\mathcal{A}(\Sigma_i)}$$

4 Explanation of Choice of Seeds for $\mathcal{A}_i$

In this section we explain the "greedy approach" to obtain initial seeds of $\mathcal{A}_i$, starting from the initial seeds of Section 3. The idea is to apply a minimum number of mutations to get the cluster variables belonging to a single $\mathcal{A}_i$.

$\mathcal{A}_1$ We start from the initial seed Σ_1 and apply the mutations $\mu_2\mu_4$ to obtain the following:

$$
\begin{tikzcd}
\mathbf{-1} \arrow[r] & 1 \arrow[r] \arrow[d] & 3 \arrow[d] & \mathbf{5} \arrow[l, shift left] \arrow[l, shift right] \\
\mathbf{-2} & 2 \arrow[l] \arrow[r] & 4 \arrow[ul] \arrow[ur] & \mathbf{6} \arrow[ll, bend left]
\end{tikzcd}
$$

Lemma 4.1 *Let X_2' and X_4' be functions obtained by applying mutations $\mu_2\mu_4$ to the initial seed Σ_1. Then we have*

$$X_2' = X_0(\Delta^{\omega_1}\Delta^{w_0\omega_1} + \Delta^{s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1}) - \Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1} - \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1}\Delta^{w_0\omega_1}$$

$$X_4' = \Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}$$

This lemma proves that the initial seed $\underline{\Sigma_1}$ is precisely the one obtained from $\mu_2\mu_4(\Sigma_1)$ by "freezing" the vertex labeled $\overline{2}$ (and renaming vertices, just for convenience of notation). In the mutated seed $\mu_2\mu_4(\Sigma_1)$ all cluster variables belong to representations of type $V_{n\omega_1}$ except for the ones corresponding to vertices $-\mathbf{2}$ and $\mathbf{6}$ which are only linked to vertex 2. Therefore if we do not allow mutation at this vertex, all the functions we shall obtain will again belong to $V_{n\omega_1}$ (for some n) and hence to $\mathcal{A}_1$.

$\mathcal{A}_2$ In this case we start with the initial seed Σ_2 and apply the mutations $\mu_2\mu_4$ to obtain:

$$
\begin{tikzcd}
\mathbf{-1} & 2 \arrow[l] \arrow[r] & 4 \arrow[dl] \arrow[dr] & \mathbf{6} \arrow[ll, bend right] \\
\mathbf{-2} \arrow[r] & 1 \arrow[r] \arrow[u] & 3 \arrow[u] & \mathbf{5} \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}
$$

Lemma 4.2 *Let X_2' and X_4' be functions obtained by applying the mutations $\mu_2\mu_4$ to the initial seed Σ_2. Then we have*

$$X_2' = (2f_1^\dagger f_2^\dagger - f_2^\dagger f_1^\dagger)X_4'$$

$$X_4' = \frac{1}{6}(f_1^\dagger)^2 f_2^\dagger \Delta^{\omega_2}$$

We will prove these lemmas in Section 7.

5 Proof of Theorem 2.1

In the next two subsections we describe the main constituents of the proof of Theorem 2.1. We treat the cases $i = 1$ and $i = 2$ separately.

5.1 Proof for Case $i = 1$

We first assume that the ground ring for the cluster algebra $\mathcal{A}(\underline{\Sigma_1})$ is $\mathbb{Z}[X_{-1}, X_{-2}, X_{-3}]$.

Lemma 4.1 implies that cluster algebra $\mathcal{A}(\underline{\Sigma_1})$ is a subalgebra of $\mathcal{A}(\Sigma_1)$, that is the functions obtained (as cluster variables) by applying mutations to $\underline{\Sigma_1}$ are cluster variables of $\mathcal{A}(\Sigma_1)$. This allows us to apply Theorem 3.1 to conclude that

$$\mathcal{A}(\underline{\Sigma_1}) \subset \mathcal{A}_1$$

We define the following $\mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}$ grading on $\mathcal{A}(\Sigma_1)$. An element $F \in \mathcal{A}(\underline{\Sigma_1})$ has degree (n, p, q) if considered as an element of $\mathcal{A}_1$, we have

$$F \in V_{n\omega_1}(p\alpha_1 + q\alpha_2)$$

With this definition we have degrees of cluster variables of initial seed $\underline{\Sigma_1}$:

$$\deg(X_{-1}) = (1, -2, -1),\ \deg(X_{-2}) = (3, 0, 0),\ \deg(X_{-3}) = (1, 2, 1)$$

$$\deg(X_1) = (1, -1, -1),\ \deg(X_2) = (3, 3, 1),\ \deg(X_3) = (1, 1, 0)$$

Recall that the first fundamental representation V_{ω_1} has the realization in terms of generalized minors (see Fig. 1).

Since the algebra $\mathcal{A}_1$ is generated by the weight vectors of V_{ω_1}, in order to prove the first part of Theorem 2.1, we need to prove that the inclusion $\mathcal{A}(\underline{\Sigma_1}) \subset \mathcal{A}_1$ becomes equality when localized at Δ^{ω_1}. Thus it would suffice to prove the following two statements.

a) There are cluster variables W, Y such that

$$\deg(W) = (1, -1, 0),\ \deg(Y) = (1, 1, 1)$$

b) $X_0\Delta^{\omega_1}$ can be written in terms of cluster variables (i.e, it belongs to the algebra generated by cluster variables).

$$\begin{tikzcd}
 & \Delta^{s_1\omega_1} \arrow[drrr, "f_2"] & & \Delta^{\omega_1} \arrow[ll, "f_1"'] & \\
\Delta^{s_1s_2s_1\omega_1} \arrow[drrr, "f_2"] & & X_0 \arrow[ll, "f_1"] & & \Delta^{s_2s_1\omega_1} \arrow[ll, "f_1/2"] \\
 & \Delta^{w_0\omega_1} & & \Delta^{s_2s_1s_2s_1\omega_1} \arrow[ll, "f_1"] &
\end{tikzcd}$$

Fig. 1 First fundamental representation

We begin by applying μ_3 to $\underline{\Sigma_1}$. Let X_3' be the cluster variable obtained at vertex 3.

$$X_3' = \frac{X_1 X_{-3}^2 + X_2}{X_3} = \Delta^{s_1\omega_1}\Delta^{s_2 s_1\omega_1} - \Delta^{\omega_1} X_0$$

Therefore we have $\Delta^{\omega_1} X_0$ in terms of other cluster variables (assuming that $\Delta^{s_1\omega_1}$ and $\Delta^{s_2 s_1 \omega_1}$ are cluster variables: the assertion of part a)).

Let us define $\underline{\Sigma_1}^0 := \mu_2\mu_3(\underline{\Sigma_1})$. The exchange matrix at this cluster is given by

$$\begin{tikzcd}
\mathbf{-1} \arrow[r] & 1 \arrow[d] & 3 \arrow[l] \arrow[d] & \mathbf{-3} \arrow[l] \\
 & \mathbf{-2} \arrow[ur] & 2 \arrow[l] \arrow[ur] &
\end{tikzcd}$$

Therefore the seed $\underline{\Sigma_1}^0$ is bipartite. Following [7] we define

$$\mu_+ = \mu_1\mu_2, \quad \mu_- = \mu_3$$

As long as we restrict ourselves to "the bipartite belt" the mutations μ_1 and μ_2 commute (since vertices 1 and 2 are not linked). Further we define

$$\underline{\Sigma_1}^r = \begin{cases} \underbrace{\cdots \mu_-\mu_+\mu_-\mu_+}_{r \text{ terms}}(\underline{\Sigma_1}^0) \ if\ r \geq 0 \\ \underbrace{\cdots \mu_+\mu_-\mu_+\mu_-}_{-r \text{ terms}}(\underline{\Sigma_1}^0) \ if\ r < 0 \end{cases}$$

We denote by $X_i^{(r)}$, the cluster variables at the seed $\underline{\Sigma_1}^r$ and $d_i^{(r)} \in \mathbb{N}\times\mathbb{Z}\times\mathbb{Z}$ its degree. Then part a) follows from the following proposition: to see that there exist cluster variables with degrees $(1,-1,0)$ and $(1,1,1)$ we just observe that $d_1^{(-3)} = (1,1,1)$ and $d_1^{(-7)} = (1,-1,0)$. This proposition will be proved in Section 6.

Proposition 5.1 *In the notation introduced above, we have*

$$d_1^{(2r)} = d_1^{(2r-1)}, \quad d_2^{(2r)} = d_2^{(2r-1)}, \quad d_3^{(2r+1)} = d_3^{(2r)}$$

$$d_1^{(2r+1)} = \begin{cases} (r/2+1, 1, 1) & if\ r \geq 0 \ and\ even \\ ((r+1)/2, 1, 0) & if\ r > 0\ and\ odd \\ (1,1,1) & if\ r = -2 \\ ((1-r)/2, -1, -1) & if\ r < 0\ and\ odd \\ (-r/2-1, -1, 0) & if\ r < -2\ and\ even \end{cases}$$

$$d_2^{(2r+1)} = \begin{cases} (3r/2+3, 3, 1) & if\ r \geq 0\ and\ even \\ (3(r+1)/2+3, 3, 2) & if\ r > 0\ and\ odd \\ (3,3,2) & if\ r = -1 \\ (-3r/2, -3, -2) & if\ r < 0\ and\ even \\ (-3(r+1)/2, -3, -1) & if\ r < -1\ and\ odd \end{cases}$$

$$d_3^{(2r)} = \begin{cases} (2+r, 2, 1) & if\ r \geq 0 \\ (2,0,0) & if\ r = -1 \\ (-r, -2, -1) & if\ r < -1 \end{cases}$$

5.2 Proof for the Case $i = 2$

The proof of this part is exactly similar to the previous one. We restrict ourselves to the ground ring $\mathbb{Z}[Y_{-1}, Y_{-2}, Y_{-3}]$. Assuming Lemma 4.2 we conclude from Theorem 3.1 the inclusion

$$\mathcal{A}(\underline{\Sigma_2}) \subset \mathcal{A}_2$$

which allows us to define a grading on elements of $\mathcal{A}(\underline{\Sigma_2})$. We say $\deg(F) = (n, p, q)$ if as an element of $\mathcal{A}_2$ we have

$$F \in V_{n\omega_2}(p\alpha_1 + q\alpha_2)$$

In this notation the cluster variables have the following degrees

$$\deg(Y_{-1}) = (1, -3, -2),\quad \deg(Y_{-2}) = (1, 0, 0),\quad \deg(Y_{-3}) = (1, 3, 2)$$

$$\deg(Y_1) = (1, -3, -1),\quad \deg(Y_2) = (1, 1, 1),\quad \deg(Y_3) = (1, 0, 1)$$

Note that the second fundamental representation of G has the realization in terms of certain functions on G (see Fig. 2).

In the figure above, each function represents a chosen basis vector of the respective weight space and the arrows indicate action of lowering operators. For instance $\Delta^{s_1 s_2 \omega_2} \xrightarrow{f_2=(1\ 2)^T} \begin{matrix} F^1(0,0) \\ F^2(0,0) \end{matrix}$ means that $f_2^\dagger(\Delta^{s_1 s_2 \omega_2}) = F^1(0,0) + 2F^2(0,0)$. The functions $F(i, j)$ have weight $\alpha_1 i + \alpha_2 j$ and Fig. 2 can be taken as definition of these functions. For example

$$F(1,1) = \frac{1}{6}(f_1^\dagger)^2 \Delta^{s_2\omega_2} = Y_2$$

$$F^1(0,0) = (2 f_1^\dagger f_2^\dagger - f_2^\dagger f_1^\dagger)(F(1,1)) = Y_{-2}$$

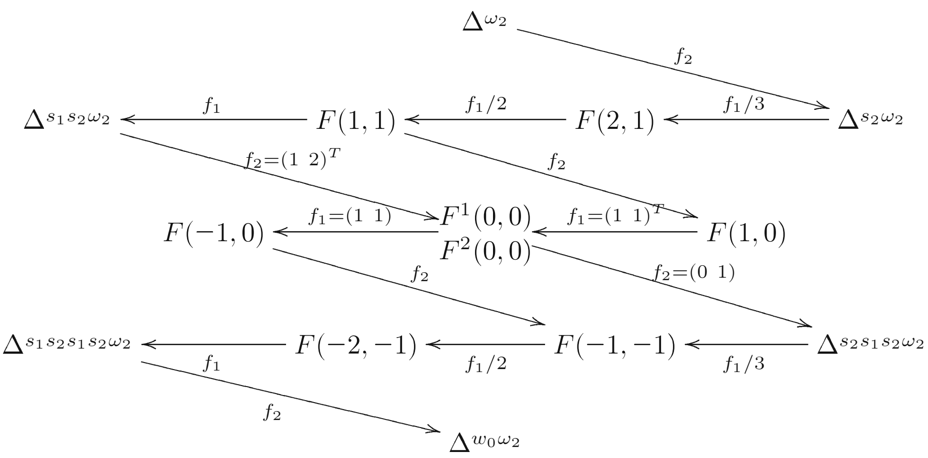

Fig. 2 Second fundamental representation

Again it suffices to prove the following two statements

a) There exist cluster variables with degrees $(1, 3, 1)$; $(1, 2, 1)$; $(1, 1, 0)$; $(1, -1, 0)$; $(1, 0, -1)$; $(1, -1, -1)$ and $(1, -2, -1)$.
b) $\Delta^{\omega_2}.F^2(0, 0)$ can be written as a polynomial in cluster variables.

Similar to the previous part, we define $\underline{\Sigma_2}^0$ to be $\mu_2\mu_3(\underline{\Sigma_2})$ to make it bipartite:

$$\begin{tikzcd}
 & \mathbf{-2} \arrow[dr] & 2 \arrow[l] \arrow[dr] & \\
\mathbf{-1} \arrow[r] & 1 \arrow[u] & 3 \arrow[l] \arrow[u] & \mathbf{-3} \arrow[l]
\end{tikzcd}$$

And define

$$\mu_+ = \mu_1\mu_2, \quad \mu_- = \mu_3$$

The bipartite belt consists of the following seeds

$$\underline{\Sigma_2}^r = \begin{cases} \underbrace{\cdots \mu_-\mu_+\mu_-\mu_+}_{r \text{ terms}}(\underline{\Sigma_2}^0) \ if\ r \geq 0 \\ \underbrace{\cdots \mu_+\mu_-\mu_+\mu_-}_{-r \text{ terms}}(\underline{\Sigma_2}^0) \ if\ r < 0 \end{cases}$$

Again we let $Y_i^{(r)}$ be the cluster variables in $\underline{\Sigma_2}^0$ and let $g_i^{(r)}$ be its degree. Also let U and Z be cluster variables appearing at vertex 2 in $\mu_2\mu_3\mu_1(\underline{\Sigma_2}^0)$ and $\mu_2\mu_3\mu_2(\underline{\Sigma_2}^0)$. Then part a) follows from the following degree computation: again we only need to observe that $g_1^{(-3)} = (1, 3, 1)$, $g_2^{(-3)} = (1, 2, 1)$, $g_1^{(-7)} = (1, 0, -1)$, $g_2^{(-5)} = (1, -1, -1)$, $g_2^{(-3)} = (1, -2, -1)$. This together with the first statement of the proposition exhausts the list of weights demanded in part a). Again the proof of this proposition is given in Section 6.

Proposition 5.2 *We have*

$$deg(U) = (1, 1, 0), \quad deg(Z) = (1, -1, 0)$$

Moreover we have

$$g_1^{(2r)} = g_1^{(2r-1)}, \quad g_2^{(2r)} = g_2^{(2r-1)}, \quad g_3^{(2r+1)} = g_3^{(2r)}$$

$$g_1^{(2r+1)} = \begin{cases} (r/2 + 2, 3, 1) & if\ r \geq 0\ and\ even \\ ((r+1)/2, 0, 1) & if\ r > 0\ and\ odd \\ (1, 3, 1) & if\ r = -2 \\ ((1-r)/2, -3, -1) & if\ r < 0\ and\ odd \\ (-r/2 - 1, 0, -1) & if\ r < -2\ and\ even \end{cases}$$

$$g_2^{(2r+1)} = \begin{cases} (r/2 + 1, 2, 1) & if\ r \geq 0\ and\ even \\ ((r+3)/2, 1, 1) & if\ r > 0\ and\ odd \\ (1, 2, 1) & if\ r = -1 \\ (-r/2, -2, -1) & if\ r < 0\ and\ even \\ (-(r+1)/2, -1, -1) & if\ r < -1\ and\ odd \end{cases}$$

$$g_3^{(2r)} = \begin{cases} (2+r, 3, 2) & \textit{if } r \geq 0 \\ (2, 0, 0) & \textit{if } r = -1 \\ (-r, -3, -2) & \textit{if } r < -1 \end{cases}$$

Part b) will follow from above proposition together with following "Plücker relation" (see Section 7.2):

$$\Delta^{\omega_2}(F^1(0,0) + F^2(0,0)) = \Delta^{s_2\omega_2}\Delta^{s_1 s_2 \omega_2} - F(1,1)F(2,1)$$

Remark 5.1 Thus to prove the main result we are reduced to proving two Lemmas (4.1 and 4.2) and two Propositions (5.1 and 5.2). Both the propositions are very combinatorial in nature and are proved in next section. It turns out that degrees $d_i^{(r)}$ and $g_i^{(r)}$ satisfy the same recurrence relations and just have different initial values, which allows us to prove them simultaneously. The lemmas stumble upon certain "determinant identities" which we prove using the representation theory of G_2 in Section 7.

Remark 5.2 It follows from the general theory of cluster algebras of affine type (the reader can consult [3] for some structural results) that U, Z and $Y_i^{(r)}$ are all the cluster variables of cluster algebra $\mathcal{A}(\underline{\Sigma_2})$ (and similarly for $\mathcal{A}(\underline{\Sigma_1})$).

6 Proofs of Propositions 5.1 and 5.2

This section is devoted to the computation of degrees. In order to state the problem at hand in purely combinatorial terms, we review the set up a little bit.

Consider the following quivers (transpose of each other):

$$\Gamma_1^0 : \begin{tikzcd} \mathbf{-1} \arrow[r] & 1 \arrow[d] & 3 \arrow[l] \arrow[d] & \mathbf{-3} \arrow[l] \\ & \mathbf{-2} \arrow[ur] & 2 \arrow[l] \arrow[ur] & \end{tikzcd}$$

$$\Gamma_2^0 : \begin{tikzcd} & \mathbf{-2} \arrow[dr] & 2 \arrow[l] \arrow[dr] & \\ \mathbf{-1} \arrow[r] & 1 \arrow[u] & 3 \arrow[l] \arrow[u] & \mathbf{-3} \arrow[l] \end{tikzcd}$$

We apply the mutation at vertices $1, 2$ or 3. Let us call arrows between vertices of same parity as *horizontal* and between different parity *vertical*. The following list describes the mutation rules for μ_i (note the appearance of factor 3 to accommodate the valuations on vertical arrows)

(M1) If $j \to i \to k$ are not both vertical, then add one arrow $j \to k$.
(M1') If $j \to i \to k$ are both vertical then add three arrows from j to k.
(M2) If $i \to j$ is an arrow (horizontal or vertical) then change it to $i \leftarrow j$ (similarly the other way around).

(M3) Finally remove all arrows between vertices $-1, -2$ and -3 and cancel two cycles, i.e, if there are s arrows from j to k and t arrows from k to j and $s \geq t$ then just write $s - t$ arrows from j to k.

It is clear that if two vertices i and j are not connected then $\mu_i \mu_j = \mu_j \mu_i$. This allows us to unambiguously define $\mu_+ = \mu_1 \mu_2$ and $\mu_- = \mu_3$. Set

$$\Gamma_i^r := \begin{cases} \underbrace{\cdots \mu_- \mu_+ \mu_- \mu_+}_{r \text{ terms}} \Gamma_i^0 \text{ if } r \geq 0 \\ \underbrace{\cdots \mu_+ \mu_- \mu_+ \mu_-}_{-r \text{ terms}} \Gamma_i^0 \text{ if } r < 0 \end{cases}$$

6.1 Structure of Graphs Γ_i^r

It is clear that between vertices $1, 2$ and 3 the above graphs have the following structure:

$$r \text{ is even} \quad \begin{tikzcd} 1 & 3 \arrow[l] \arrow[d] \\ & 2 \end{tikzcd} \qquad r \text{ is odd} \quad \begin{tikzcd} 1 \arrow[r] & 3 \\ & 2 \arrow[u] \end{tikzcd}$$

Since there are no arrows between the vertices $-1, -2$ and -3, in order to completely determine the structure of the graphs Γ_i^r we need to compute the following matrix

$$C_{ij}^{(r)} := \text{ Number of arrows from } i \text{ to } j$$

where $i \in \{-1, -2, -3\}$ and $j \in \{1, 2, 3\}$. It is clear that both the graphs Γ_1 and Γ_2 have the same C-matrix.

The mutation rules define the following recurrence relations among entries of $C^{(r)}$ (where the notation $[x]_- := [-x]_+ = \max(0, -x)$ is used):

$$C^{(r+1)} = \begin{cases} \begin{pmatrix} -C_{-1,1}^{(r)} & -C_{-1,2}^{(r)} & C_{-1,3}^{(r)} - [C_{-1,1}^{(r)}]_- - 3[C_{-1,2}^{(r)}]_- \\ -C_{-2,1}^{(r)} & -C_{-2,2}^{(r)} & C_{-2,3}^{(r)} - [C_{-2,1}^{(r)}]_- - [C_{-2,2}^{(r)}]_- \\ -C_{-3,1}^{(r)} & -C_{-3,2}^{(r)} & C_{-3,3}^{(r)} - [C_{-3,1}^{(r)}]_- - 3[C_{-3,2}^{(r)}]_- \end{pmatrix} & \text{if } r \text{ is even} \\ \\ \begin{pmatrix} C_{-1,1}^{(r)} - [C_{-1,3}^{(r)}]_- & C_{-1,2}^{(r)} - [C_{-1,3}^{(r)}]_- & -C_{-1,3}^{(r)} \\ C_{-2,1}^{(r)} - [C_{-2,3}^{(r)}]_- & C_{-2,2}^{(r)} - 3[C_{-2,3}^{(r)}]_- & -C_{-2,3}^{(r)} \\ C_{-3,1}^{(r)} - [C_{-3,3}^{(r)}]_- & C_{-3,2}^{(r)} - [C_{-3,3}^{(r)}]_- & -C_{-3,3}^{(r)} \end{pmatrix} & \text{if } r \text{ is odd} \end{cases}$$

with the initial value

$$C^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

One can verify directly that the following lemma gives the solution of this recurrence relation:

Lemma 6.1 *Let $C_i^{(r)}$ be the row of C labeled by i. Then we have the following solution*

$$C_{-1}^{(r)} = -C_{-1}^{(-r+1)}$$

$$C_{-1}^{(r)} = \left(C_{-1,1}^{(r)}, (-1)^{r+1}\left\lfloor \frac{r}{4} \right\rfloor, (-1)^r \left(\left\lceil \frac{r}{2} \right\rceil - 1\right)\right) \ \textit{for}\ r \geq 1$$

$$C_{-1,1}^{(2r)} = C_{-1,1}^{(2r+1)},\ \ C_{-1,1}^{(1)} = -1,\ \textit{and}$$

$$C_{-1,1}^{(2r+1)} = \begin{cases} \frac{r+1}{2} & r \textit{ is odd} \\ \frac{r-2}{2} & r \textit{ is even} \end{cases}$$

$$C_{-2}^{(r)} = \begin{cases} (-1, -1, 1) & r \equiv 0 \textit{ mod } 4 \\ (1, 1, -1) & r \equiv 1 \textit{ mod } 4 \\ (0, -2, 1) & r \equiv 2 \textit{ mod } 4 \\ (0, 2, -1) & r \equiv 3 \textit{ mod } 4 \end{cases}$$

$$C_{-3}^{(r)} = C_{-1}^{(r+4)}$$

6.2 Computation of Degrees

Now we are in position to prove the Propositions 5.1 and 5.2. The Proposition 5.1 reduces to an easy check (using formulae for $C'_{ij}s$ from the previous section) that the degrees stated satisfy the following recurrence relations

$$d_1^{(r+1)} = \begin{cases} d_1^{(r)} & r \text{ is odd} \\ -d_1^{(r)} + [C_{-1,1}^{(r)}]_-(1, -2, -1) + [C_{-2,1}^{(r)}]_-(3, 0, 0) + [C_{-3,1}^{(r)}]_-(1, 2, 1) & r \text{ is even} \end{cases}$$

$$d_2^{(r+1)} = \begin{cases} d_2^{(r)} & r \text{ is odd} \\ -d_2^{(r)} + 3[C_{-1,2}^{(r)}]_-(1, -2, -1) + [C_{-2,1}^{(r)}]_-(3, 0, 0) + 3[C_{-3,2}^{(r)}]_-(1, 2, 1) & r \text{ is even} \end{cases}$$

$$d_3^{(r+1)} = \begin{cases} d_3^{(r)} & r \text{ is even} \\ -d_3^{(r)} + [C_{-1,3}^{(r)}]_-(1, -2, -1) + [C_{-2,3}^{(r)}]_-(3, 00) + [C_{-3,3}^{(r)}]_-(1, 2, 1) & r \text{ is odd} \end{cases}$$

together with initial values $d_1^{(0)} = (1, -1, -1)$, $d_2^{(0)} = (2, 2, 1)$ and $d_3^{(0)} = (3, 3, 2)$.

Similarly Proposition 5.2 reduces to checking that the degrees stated satisfy the following recurrence relations:

$$g_1^{(r+1)} = \begin{cases} g_1^{(r)} & r \text{ is odd} \\ -g_1^{(r)} + [C_{-1,1}^{(r)}]_-(1, -3, -2) + 3[C_{-2,1}^{(r)}]_-(1, 0, 0) + [C_{-3,1}^{(r)}]_-(1, 3, 2) & r \text{ is even} \end{cases}$$

$$g_2^{(r+1)} = \begin{cases} g_2^{(r)} & r \text{ is odd} \\ -g_2^{(r)} + [C_{-1,2}^{(r)}]_-(1,-3,-2) + [C_{-2,1}^{(r)}]_-(1,0,0) + [C_{-3,2}^{(r)}]_-(1,3,2) & r \text{ is even} \end{cases}$$

$$g_3^{(r+1)} = \begin{cases} g_3^{(r)} & r \text{ is even} \\ -g_3^{(r)} + [C_{-1,3}^{(r)}]_-(1,-3,-2) + 3[C_{-2,3}^{(r)}]_-(1,0,0) + [C_{-3,3}^{(r)}]_-(1,3,2) & r \text{ is odd} \end{cases}$$

together with the initial condition $g_1^{(0)} = (1,-3,-1)$, $g_2^{(0)} = (1,2,1)$ and $g_3^{(0)} = (2,3,2)$.

7 Some Determinant Identities for G_2

The aim of this part is to prove certain identities among the weight vectors of V_{ω_1} and V_{ω_2}.

7.1 Determinant Identities

The Proof of Lemmas 4.1 and 4.2 follows from the following identities:

(D1)

$$\begin{aligned} &\Delta^{\omega_2}(\Delta^{s_2s_1\omega_1})^3 + (\Delta^{\omega_1})^3\Delta^{s_2s_1s_2\omega_2} \\ &\quad = \Delta^{s_2\omega_2}\left(\Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}\right) \end{aligned}$$

(D2)

$$\begin{aligned} &(\Delta^{s_2s_1s_2s_1\omega_1})^3\Delta^{\omega_2} + \Delta^{w_0\omega_2}\left(\Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}\right) \\ &\quad = \Delta^{s_2s_1s_2\omega_2}\,(X_0(\Delta^{\omega_1}\Delta^{w_0\omega_1} + \Delta^{s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1}) \\ &\qquad -\Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1} - \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1}\Delta^{w_0\omega_1}) \end{aligned}$$

(D3)

$$\Delta^{s_1s_2s_1\omega_1}\Delta^{\omega_2} + \Delta^{\omega_1}\Delta^{s_1s_2\omega_2} = \Delta^{s_1\omega_1}F(1,1)$$

(D4)

$$\Delta^{\omega_1}\Delta^{s_1s_2s_1s_2\omega_2} + \Delta^{w_0\omega_1}F(1,1) = \Delta^{s_1s_2s_1\omega_1}F^1(0,0)$$

In order to prove these identities, one uses the fact that if $F \in V_\lambda(\mu)$ and $\mu < \lambda$, then $F = 0$ if and only if $e_iF = 0$ for $i = 1, 2$ (and using the action of operators $e_i's$ from Figs. 1 and 2).

For instance, consider $\xi = \Delta^{\omega_1}F(2,1) - \Delta^{s_1\omega_1}\Delta^{s_2\omega_2} + \Delta^{s_2s_1\omega_1}\Delta^{\omega_2}$. It is easy to compute and verify that $e_1\xi = e_2\xi = 0$ and hence $\xi = 0$. More relations of this kind can be obtained by applying f_1, f_2 to ξ (there are in fact 35 relations of this type, out of which 27 (=$dim(V_{2\omega_1})$) can be obtained from ξ). The relations (D1)–(D4) can be checked by direct calculation (for instance, (D3) is in fact obtained by applying $(f_1)^2$ to the equality $\xi = 0$).

7.2 Plücker Relations

We will also need the following "Plücker relations" which hold among the weight vectors of V_{ω_2}:

(P1)

$$F(2,1)^2 + \Delta^{\omega_2} F(1,0) = \Delta^{s_2\omega_2} F(1,1)$$

(P2)

$$F(1,0)\Delta^{s_1s_2s_1s_2\omega_2} = F^1(0,0)F(-2,-1) + F(1,1)\Delta^{w_0\omega_2}$$

(P3)

$$\Delta^{\omega_2}\Delta^{w_0\omega_2} = F(1,0)F(-1,0) - F^1(0,0)F^2(0,0)$$

(P4)

$$F(-2,-1)F(2,1) + F(1,0)F(-1,0) = F(1,1)F(-1,-1)$$

(P5)

$$F(1,1)^2 = F(2,1)\Delta^{s_1s_2\omega_2} - \Delta^{\omega_2} F(-1,0)$$

(P6)

$$\Delta^{\omega_2}\Delta^{s_1s_2s_1s_2\omega_2} = F(1,1)F(-1,0) - F^1(0,0)\Delta^{s_1s_2\omega_2}$$

The proof of these relations is exactly the same as the one in the previous section. For example, the relation (P1) is equivalent to: $e_i^\dagger(F(2,1)^2 + \Delta^{\omega_1} F(1,0) - \Delta^{s_2\omega_2} F(1,1)) = 0$ for $i = 1, 2$, which can be verified directly using the action of operators $e_i^\dagger$ from Fig. 2. Other relations are obtained by applying lowering operators to both sides of (P1). For instance, (P5) can be obtained by applying $(1/4)(f_1^\dagger)^2$ to (P1).

8 Comparison with Initial Seed of [8]

In [8], the authors give a uniform combinatorial description of the initial seeds for partial flag varieties. In this section we describe their construction for the case at hand. The algorithm in [8] first constructs the seed for $\mathbb{C}[N_i]$ (where N_i is unipotent radical of P_i for $i = 1, 2$) and then lift it to a seed of $\mathbb{C}[P_j^- \backslash G]$ ($j = \{1, 2\} \setminus \{i\}$). In the next two paragraphs we will give this lifting. The reader can refer to [8] for details.

8.1 Case $i = 1$

In this case the initial seed for $\mathbb{C}[P_2^- \backslash G]$, denoted by $\Sigma_1^{(GLS)}$ has exchange matrix given by

$$\begin{tikzcd}
\Gamma_1^{(GLS)}: & \mathbf{-1} \arrow[r] & 1 \arrow[r] \arrow[d] & 3 \arrow[d] & \mathbf{-3} \arrow[l] \\
 & & \mathbf{-2} \arrow[r] & 2 \arrow[ul] &
\end{tikzcd}$$

with corresponding functions given by

$$X_{-1}^{(GLS)} = \Delta^{w_0\omega_1},\ X_{-3}^{(GLS)} = \Delta^{\omega_1}$$

$$X_{-2}^{(GLS)} = X_0(\Delta^{\omega_1}\Delta^{w_0\omega_1} + \Delta^{s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1}) - \Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1}\Delta^{s_2s_1s_2s_1\omega_1} - \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1}\Delta^{w_0\omega_1}$$

$$X_1^{(GLS)} = -\Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1} + \Delta^{s_1\omega_1}X_0$$

$$X_2^{(GLS)} = \Delta^{s_2s_1\omega_1}(\Delta^{s_1\omega_1})^2 - 2\Delta^{\omega_1}\Delta^{s_1\omega_1}X_0 + (\Delta^{\omega_1})^2\Delta^{s_1s_2s_1\omega_1},\ X_3^{(GLS)} = \Delta^{s_1\omega_1}$$

Claim The sequence of mutations $\mu_1\mu_3$ applied to $\Sigma_1^{(GLS)}$ produces the same cluster as $\underline{\Sigma_1}^0$ with exchange matrix $-\underline{\widetilde{B}_1}^0$.

Since $\underline{\Sigma_1}^0$ is obtained from the initial seed $\underline{\Sigma_1}$ by applying the mutations $\mu_2\mu_3$, in order to prove the claim we need to compute $\mu_3\mu_2\mu_1\mu_3(\Sigma_1^{(GLS)})$. We do this computation in following steps:

a) $\mu_3(\Gamma_1^{(GLS)})$ is:

$$\begin{tikzcd}
\mathbf{-1} \arrow[r] & 1 \arrow[d] & 3 \arrow[l] \arrow[r] & \mathbf{-3} \arrow[dl] \\
 & \mathbf{-2} \arrow[r] & 2 \arrow[u] &
\end{tikzcd}$$

And the cluster variable at third vertex mutates as:

$$X_3' = \frac{\Delta^{\omega_1}(\Delta^{s_1\omega_1}X_0 - \Delta^{\omega_1}\Delta^{s_1s_2s_1\omega_1}) + (\Delta^{s_1\omega_1})^2\Delta^{s_2s_1\omega_1} - 2\Delta^{s_1\omega_1}\Delta^{\omega_1}X_0 + (\Delta^{\omega_1})^2\Delta^{s_1s_2s_1\omega_1}}{\Delta^{s_1\omega_1}}$$

$$X_3' = \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1} - \Delta^{\omega_1}X_0$$

b) $\mu_1\mu_3(\Gamma_1^{(GLS)})$ is:

$$\begin{tikzcd}
\mathbf{-1} & 1 \arrow[l] \arrow[r] & 3 \arrow[r] \arrow[dl] & \mathbf{-3} \arrow[dl] \\
 & \mathbf{-2} \arrow[u] \arrow[r] & 2 \arrow[u] &
\end{tikzcd}$$

And the cluster variable at vertex 1 mutates as:

$$X_1' = \frac{X_{-1}X_3' + X_{-2}}{X_1}$$

This exchange relation gives $X_1' = \Delta^{s_2s_1s_2s_1\omega_1}$.

c) $\mu_2\mu_1\mu_3(\Sigma_1^{(GLS)})$ is:

$$\begin{tikzcd}
\mathbf{-1} & 1 \arrow[l] \arrow[r] & 3 \arrow[d] & \mathbf{-3} \arrow[l, Rightarrow] \\
& \mathbf{-2} \arrow[u] & 2 \arrow[l] \arrow[ur] &
\end{tikzcd}$$

The new cluster variable at vertex 2 is obtained as:

$$X_2' = \frac{(X_3')^3 + X_{-2}X_{-1}}{X_2}$$

This can be easily computed to be $X_2' = \Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}$.

d) Finally $\mu_3\mu_2\mu_1\mu_3(\Gamma_1^{(GLS)})$ is:

$$\begin{tikzcd}
\mathbf{-1} & 1 \arrow[l] \arrow[dr] & 3 \arrow[l] \arrow[r, Rightarrow] & \mathbf{-3} \arrow[dl] \\
& \mathbf{-2} \arrow[u] & 2 \arrow[l] \arrow[u] &
\end{tikzcd}$$

which is the same as our initial exchange matrix except for the fact that all arrows are reversed. And the last cluster variable here (at vertex 3) is:

$$X_3'' = \frac{(\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1} + \Delta^{s_1\omega_1}(\Delta^{s_2s_1\omega_1})^2 - \Delta^{\omega_1}\Delta^{s_2s_1\omega_1}X_0 - (\Delta^{\omega_1})^2\Delta^{s_2s_1s_2s_1\omega_1}}{\Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1} - \Delta^{\omega_1}X_0}$$

Therefore $X_3'' = \Delta^{s_2s_1\omega_1}$.

Thus we have shown that our initial seed Σ_1 can be obtained from the one given in [8] by a sequence of mutations (up to a sign). Since the cluster algebras associated to the matrices $\widetilde{B}$ and $-\widetilde{B}$ are naturally isomorphic, both initial seeds give rise to the same cluster algebra.

8.2 Case $i = 2$

We start with the reduced word $\mathbf{i_1} = (1, 2, 1, 2, 1, 2)$ of w_0. The initial seed for $\mathbb{C}[P_1^- \backslash G]$, denoted by $\Sigma_2^{(GLS)}$ has exchange matrix given by:

$$\Gamma_2^{(GLS)}: \quad \begin{tikzcd}
& \mathbf{-2} \arrow[r] & 2 \arrow[dl] & \\
\mathbf{-1} \arrow[r] & 1 \arrow[u] \arrow[r] & 3 \arrow[u] & \mathbf{-3} \arrow[l]
\end{tikzcd}$$

with corresponding functions given by

$$Y_{-1}^{(GLS)} = \Delta^{w_0\omega_2},\ Y_{-2}^{(GLS)} = F^1(0,0),\ Y_{-3}^{(GLS)} = \Delta^{\omega_2}$$

$$Y_1^{(GLS)} = F(2,1)F(1,0) - \Delta^{s_2\omega_2}F^1(0,0),\ Y_2^{(GLS)} = F(2,1),\ Y_3^{(GLS)} = \Delta^{s_2\omega_2}$$

Claim The sequence of mutations $\mu_1\mu_3$ applied to $\Sigma_2^{(GLS)}$ yields the cluster of $\underline{\Sigma_2}^0$ with exchange matrix multiplied by -1.

The proof of the analogous claim from the last paragraph carries verbatim over to a proof of this claim. One only needs to make use of the "Plücker Relations" ((P1)–(P6)) from last section in order to carry out the computations involving exchange relations.

9 Summary of Results

In this section we give a brief summary of the results proved in this note.

9.1 Case $i = 1$

- The initial seed $\underline{\Sigma_1}$ can be obtained from the initial seed Σ_1 of $\mathcal{A}$ by applying the mutations $\mu_2\mu_4$ and freezing vertex 2 (see Section 4).
- The cluster algebra associated to the initial seed $\underline{\Sigma_1}$ is isomorphic to the algebra $\mathcal{A}_1$ localized at the multiplicative set $\{(\Delta^{\omega_1})^n\}_{n\in\mathbb{Z}_{\geq 0}}$. Explicitly we obtain all the weight vectors of Fig. 1 by applying mutations to $\underline{\Sigma_1}$, as follows (see Section 5.1):

$$\Delta^{\omega_1} = X_{-3},\quad \Delta^{w_0\omega_1} = X_{-1},\quad \Delta^{s_2s_1\omega_1} = X_3,\quad \Delta^{s_2s_1s_2s_1\omega_1} = X_1$$

$$\Delta^{s_1\omega_1} = X_1^{(-2)} = \text{ function at vertex 1 of } \mu_1\mu_3\mu_2\mu_3(\underline{\Sigma_1})$$

$$\Delta^{s_1s_2s_1\omega_1} = X_1^{(-6)} = \text{ function at vertex 1 of } \mu_1\mu_3(\mu_1\mu_2\mu_3)^2\mu_2\mu_3(\underline{\Sigma_1})$$

  And finally we have $\Delta^{\omega_1}X_0 = \Delta^{s_1\omega_1}\Delta^{s_2s_1\omega_1} - X_3^{(0)}$.
- The initial seed $\underline{\Sigma_1}$ can also be obtained by applying the sequence of mutations $\mu_3\mu_2\mu_1\mu_3$ to the initial seed $\Sigma_1^{(GLS)}$ as given in [8] (see Section 8).

9.2 Case $i = 2$

- The initial seed $\underline{\Sigma_2}$ can be obtained from the initial seed Σ_2 of $\mathcal{A}$ by applying the mutations $\mu_2\mu_4$ and freezing vertex 2 (Section 4).
- The cluster algebra associated to the initial seed $\underline{\Sigma_2}$ is isomorphic to $\mathcal{A}_2$ localized at the multiplicative set $\{(\Delta^{\omega_2})^n\}_{n\in\mathbb{Z}_{\geq 0}}$. Explicitly we obtain the weight vectors of Fig. 2 as (see Section 5.2):

$$\Delta^{\omega_2} = Y_{-3},\quad \Delta^{w_0\omega_2} = Y_{-1},\quad F^1(0,0) = Y_{-2}$$

$$\Delta^{s_1s_2s_1s_2\omega_2} = Y_1,\quad F(1,1) = Y_2,\quad \Delta^{s_1s_2\omega_2} = Y_3$$

$$\Delta^{s_2\omega_2} = Y_1^{(-2)}$$

$$F(2,1) = Y_2^{(0)}$$

$$\Delta^{s_2 s_1 s_2 \omega_2} = Y_1^{(-6)}$$

$$F(-1,-1) = Y_2^{(-4)}$$

$$F(-2,-1) = Y_2^{(-2)}$$

F(1,0) and F$(-1,0)$ are functions at vertex 2 of $\mu_2\mu_3\mu_1(\underline{\Sigma_2}^0)$ and $\mu_2\mu_3\mu_2(\underline{\Sigma_2}^0)$ respectively

Finally we have the following Plücker relation:

$$F^2(0,0)\Delta^{\omega_2} = \Delta^{s_2\omega_2}\Delta^{s_1 s_2 \omega_2} - F(1,1)F(2,1) - \Delta^{\omega_2}F^1(0,0)$$

– The initial seed $\underline{\Sigma_2}$ can also be obtained by applying $\mu_3\mu_2\mu_1\mu_3$ to the initial seed $\Sigma_2^{(GLS)}$ (see Section 8).

Acknowledgements I would like to thank Prof. Andrei Zelevinsky for introducing me to the theory of cluster algebras and for several helpful and stimulating discussions. I am also grateful to Prof. Bernard Leclerc for kindly explaining to me with the construction of initial seeds, as in [8].

References

1. Berenstein, A., Zelevinsky, A.: Total positivity in Schubert varieties. Comment. Math. Helv. **72**(1), 128–166 (1997)
2. Berenstein, A., Fomin, S., Zelevinsky, A.: Cluster algebras III: upper bounds and double Bruhat cells. Duke Math. J. **126**(1), 1–52 (2005)
3. Irelli, G.C.: Structural theory of rank three cluster algebras of affine type. Ph.D. Thesis at Università degli studi di Padova, Italy (2008)
4. Demonet, L.: Cluster algebras and preprojective algebras: the non simply-laced case. arxiv:0711.4098v1 (2008)
5. Fomin, S., Zelevinsky, A.: Double Bruhat cells and total positivity. J. Am. Math. Soc. **12**(2), 335–380 (1999)
6. Fomin, S., Zelevinsky, A.: Cluster algebras I: foundations. J. Am. Math. Soc. **15**(2), 497–529 (2002)
7. Fomin, S., Zelevinsky, A.: Cluster algebras IV: coefficients. Compos. Math. **143**, 112–164 (2007)
8. Geiss, C., Leclerc, B., Schröer, J.: Partial flag varieties and preprojective algebras. arXiv:math/0609138 (2008)